SOME NEW HILBERT ALGEBRAS

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1. **Introduction.** The object of the present paper is the introduction of a new class of commutative Hilbert algebras. These algebras are distinct from those produced by a number of variations of the original axioms of an H^*-algebra given by W. Ambrose [1], [2]. The structure theorems obtained in the previous modifications give decompositions of the algebra into orthogonal subspaces, each of which is a minimal left ideal. In the commutative case, these result in an orthogonal basis for the space consisting of minimal, usually selfadjoint, idempotents. By adopting a different set of axioms, the present author determines a more general set of commutative H-algebras in which the direct sum decomposition is not necessarily an orthogonal one. The set $\{e_i\}$ of minimal idempotents which appear need neither be selfadjoint nor orthogonal, but rather make up the elements of a Fischer-Riesz system, a concept introduced by N. Bary [3]. A brief summary of the theory of Bary's is given in §4.

This paper obtains structure theorems for three different formulations of the axioms. §2 discusses the concept of a *regular algebra* in which the fundamental assumption is that every maximal modular ideal R have a complementary ideal. §3 introduces the concepts of an adjoint algebra and of a dual adjoint algebra. An *adjoint algebra* is an algebra with two binary operations which permit the appearance of nonselfadjoint idempotents in the commutative case. A *dual adjoint algebra* is an algebra with two inner products which again permit the occurrence of nonselfadjoint idempotents. §4 gives a brief summary of the concepts of Bessel system, Hilbert system, and Fischer-Riesz system introduced by N. Bary. It also contains a basic example of a more general type of H-algebra than the standard H^*-algebra.

The symbols $\{e_i\}$, $\{g_i\}$, $(i=1, 2, \dots)$, are used to denote the elements of a biorthogonal system in a Hilbert space H. The symbols (e_i) and $[e_i]$ denote the linear hull and the closed linear hull respectively of the set of vectors $\{e_i\}$, $(i=1, 2, \dots)$. All of the algebras which appear in this paper are semisimple, commutative H-algebras, although this condition is not always expressly stated. The continuity in multiplication is given by the existence of a constant K such that

$$\|xy\| \leqq K\|x\| \, \|y\|.$$

All of the Hilbert spaces which appear as underlying spaces of the algebras are assumed to be separable. This convention is used primarily for notational convenience.

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2. **Complemented algebras.** All of the Banach algebras considered in §2 are semisimple, commutative H-algebras. An algebra A is *regular* if every maximal modular ideal R of A is complemented, i.e., there exists an ideal I of A such that A is the direct sum $R+I$.

LEMMA 2.1. *Every maximal modular ideal R of a regular algebra A has associated with it a unique minimal idempotent e and a unique multiplicative element g. An element $x \in A$ is in R if and only if either ex is the zero vector or (x, g) is the zero complex number. For every pair $x, y \in A$,*

$$(xy, g) = (x, g)(y, g), \qquad \textit{and} \qquad ex = (x, g)e. \tag{2.01}$$

Proof. The difference algebra A/R is isomorphic to the complex numbers so that the natural mapping n of A onto A/R can be regarded as a linear functional on A. Since R is closed, n is a bounded linear functional. Thus there exists a unique element $g \in A$ so that $n(x)$ and (x, g) are equal for every $x \in A$. Since n is a homomorphism of H,

$$(xy, g) = n(xy) = n(x)n(y) = (x, g)(y, g) \tag{2.02}$$

for every $x, y \in A$. This unique g is called the *associated multiplicative element* of R. Note that if e is any idempotent, then

$$(e, g) = (e^2, g) = (e, g)^2 \tag{2.03}$$

so that (e, g) must be either one or zero.

There exists at least one ideal I complementary to R. Since I is isomorphic as a vector space to A/R, it must be a one-dimensional minimal ideal of A. The idempotent generator e of I is a minimal idempotent of A. If I and I' are distinct complementary ideals of R with idempotent generators e and e' respectively, the product ee' must be zero. Consequently,

$$0 = (ee', g) = (e, g)(e', g).$$

It follows that either e or e' must be contained in R, a contradiction. Therefore the complementary minimal ideal I and the minimal idempotent generator e are uniquely determined by the maximal modular ideal R.

Note that ex must be a multiple πe of e for any $x \in A$. Since (e, g) is one, we see that

$$ex = (x, g)e. \tag{2.04}$$

A slight variation of the above argument gives the

LEMMA 2.2. *Every minimal idempotent e of a regular algebra A is the idempotent associated with some maximal regular ideal R of A.*

We emphasize the useful fact that ex is zero for a minimal idempotent e if and only if (x, g) is zero for the associated multiplicative element g, i.e., if and only if x is an element belonging to the corresponding maximal regular ideal R.

Lemma 2.3. *Let* $\{R_\pi\}$, $\{e_\pi\}$, $\{g_\pi\}$, $(\pi \in P)$, *denote the set of all maximal modular ideals, associated minimal idempotents, and associated multiplicative elements respectively of a regular algebra A. Then*

$$(2.05)\qquad \begin{aligned}(e_\pi, g_{\pi'}) &= 1, \qquad \pi = \pi',\\ &= 0, \qquad \pi \neq \pi'.\end{aligned}$$

This lemma is merely a restatement of the preceding two. Nevertheless, it points up the important fact that the two systems $\{g_\pi\}$ and $\{e_\pi\}$ form a biorthogonal system for a regular algebra A. It follows immediately from semisimplicity that the system $\{g_\pi\}$ is complete.

Lemma 2.4. *A regular algebra A contains at most a countably infinite number of distinct maximal regular ideals whenever A is a separable Hilbert space.*

Proof. The set M of multiplicative elements $\{g_\pi\}$ is a part of a biorthogonal system. By a result of S. Levin [7], no element g of M is contained in the closed linear hull $[g']$, $(g' \in M, g' \neq g)$. Let the elements of M be labelled with the members of a well-ordered set P, i.e., let

$$M = \{g_\pi : \pi \in P\}.$$

By induction, there exists an orthonormal set $\{u_\pi : \pi \in P\}$ such that for every $\pi' \in P$

$$\{[g_\pi] : \pi < \pi'\} = \{[u_\pi] : \pi < \pi'\}.$$

The set $\{u_\pi : \pi \in P\}$ must be countable which establishes the lemma.

Lemma 2.5. *Every nonzero ideal I of a regular algebra A contains a minimal idempotent.*

Proof. If I contains no minimal idempotent e_i; then $e_i x$ must be zero for every minimal idempotent e_i and every x in I. It follows that I is the zero ideal, a contradiction.

Lemma 2.6. *Every finite-dimensional ideal I of a regular algebra A is complemented, i.e., there exists a closed ideal J such that A is the direct sum* $I+J$.

Proof. Observe that every minimal idempotent e of A either belongs to I or else ex is zero for every x of I. Since the set E of minimal orthogonal idempotents of A is finitely linearly independent; only a finite set, say $\{e_i\}$, $(i=1, \dots, n)$, of minimal idempotents is contained in I. Denote by E_n the idempotent $e_1 + \cdots + e_n$ of I. Every x in A can be written

$$(2.06)\qquad x = E_n x + (x - E_n x).$$

If y denotes $x - E_n x$, then $e_i y$ is zero for every e_i contained in I. It follows that when x belongs to I, $e_i y$ vanishes for every e_i in A, implying that y is zero. Hence every x of I can be written

$$(2.07)\qquad x = E_n x = (x, g_1)e_1 + \cdots + (x, g_n)e_n.$$

Thus I is the linear hull (e_i), $(e_i \in I)$. The set of all y such that $e_i y$ is zero for e_i in I is a closed ideal J containing all elements of the form $x - E_n x$ where x is any element of A. It follows that A is the direct sum $I+J$.

In the case of H^*-algebras, orthogonality enables one to extend this lemma to any proper closed ideal of A. We recall that the *socle* of any commutative algebra A, Banach or not, is the algebraic sum of the minimal ideals of A.

LEMMA 2.7. *The socle of a regular algebra A is dense if and only if every proper closed ideal I of A is contained in a maximal regular ideal R of A.*

Proof. Suppose the socle (e_i), $(i=1, 2, \ldots)$, is dense in A. If I is a proper closed ideal, then there exists a minimal idempotent e_i not contained in I. Consequently, $e_i I$ is the zero ideal and I is contained in R_i. Let every proper ideal I of A be contained in a maximal regular ideal R of A. Then since $[e_i]$, $(i=1, 2, \ldots)$, is a closed ideal not contained in any maximal regular ideal R, it must coincide with A.

The structural analysis of H^*-algebras proceeds via the route of orthogonal, complete reducibility. At each stage, one is presented with a direct sum decomposition $I+J$ where $x \in I$, $y \in J$ imply that (x, y) is zero. Any Hilbert algebra A is said to be *well separated* if, whenever A is the direct sum $I+J$ of closed ideals I and J, there exists a constant k, with $0<k<1$, such that $x \in I$, $y \in J$ imply

$$|(x, y)| \leqq k\|x\| \, \|y\|. \tag{2.08}$$

THEOREM 2.8. *Let A be a well-separated, regular algebra in which every proper closed ideal is contained in a maximal regular ideal. Then there exists a basis of minimal idempotents, $\{e_i\}$, of A and two sequences of positive numbers, $\{d_i\}$ and $\{D_i\}$, $(i=1, 2, \ldots)$, such that*

$$\begin{aligned} d_1|x_1|^2 + \cdots + d_n|x_n|^2 + \cdots &\leqq \|x_1 e_1 + \cdots + x_n e_n + \cdots \|^2 \\ &\leqq D_1|x_1|^2 + \cdots + D_n|x_n|^2 + \cdots \end{aligned}$$

for every element $x_1 e_1 + \cdots + x_n e_n + \cdots$ of A. Furthermore, there exists a continuous homomorphism of an H^*-algebra H_1 into a dense subset of A and a continuous homomorphism of A into an H^*-algebra H_2.

Proof. Denote by E the set of all minimal idempotents of A. Let $\{u_i\}$ be an orthonormal set of A such that

$$\{(e_i), i = 1, \ldots, m\} = \{(u_i), i = 1, \ldots, m\} \tag{2.09}$$

for every m. Since the socle is dense, it follows that the set $\{u_i\}$ is an orthonormal basis of A.

For each positive integer n, let I_n and E_n denote the ideal (e_i), $(i=1, \ldots, n)$, and the idempotent $e_1 + \cdots + e_n$ respectively. According to Lemma 2.6, A is a direct sum $I_n + J_n$ of closed ideals. Consequently, any $z \in A$ can be written $x+y$ with $x \in I_n$, $y \in J_n$. It follows that

$$\|z\|^2 \geqq (1-k)(\|x\|^2 + \|y\|^2).$$

Also

$$\|z\|^2 \leqq (1+k)(\|x\|^2+\|y\|^2).$$

Finally,

$$a(\|x\|^2+\|y\|^2) \leqq \|z\|^2 \leqq b(\|x\|^2+\|y\|^2) \tag{2.10}$$

where $0<1-k=a<1<1+k=b$. If $c^2(1-k)$ is one, then

$$\|E_n z\| \leqq c\|z\|. \tag{2.11}$$

Thus the projections of the set $\{E_i\}$, $(i=1, 2, \ldots)$, are uniformly bounded by the number c.

Now, given any x in A and any positive number ε, there exists an integer N such that if

$$x_n = (x, u_1)u_1 + \cdots + (x, u_n)u_n,$$

then

$$\|x-x_n\| < \varepsilon/(1+c)$$

for n greater than N. Note that equation (2.09) implies that $x_n \in I_r$ whenever r exceeds n. Now for $N<n<r$, we have

$$\begin{aligned} \|x-E_r x\| &\leqq \|x-x_n\| + \|x_n - E_r x\| \\ &\leqq \|x-x_n\| + \|E_r(x-x_n)\| \\ &\leqq \|x-x_n\| + c\|x-x_n\| \\ &< \varepsilon. \end{aligned} \tag{2.12}$$

Thus the sequence $\{E_j x\}$, $(j=1, 2, \ldots)$, where

$$E_j x = (x, g_1)e_1 + \cdots + (x, g_j)e_j \tag{2.13}$$

converges to x for every $x \in A$. Consequently, the set E of minimal idempotents is a basis of A.

Repeated applications of inequality (2.10) lead to the following inequality,

$$\begin{aligned} |x_1|^2 a\|e_1\|^2 + \cdots + |x_n|^2 a^n \|e_n\|^2 &\leqq \|x_1 e_1 + \cdots + x_n e_n\|^2 \\ &\leqq |x_1|^2 b\|e_1\|^2 + \cdots + |x_n|^2 b^n \|e_n\|^2. \end{aligned} \tag{2.14}$$

Define two new sequences, $\{d_i\}$ and $\{D_i\}$, by means of

$$d_i = a^i\|e_i\|^2, \quad D_i = b^i\|e_i\|^2, \quad (i = 1, 2, \ldots). \tag{2.15}$$

Equation (2.14) asserts that given a sequence $\{x_n\}$ such that

$$D_1|x_1|^2 + \cdots + D_n|x_n|^2 + \cdots \tag{2.16}$$

converges, then there exists an x in A given by

$$x = x_1 e_1 + \cdots + x_n e_n + \cdots. \tag{2.17}$$

Furthermore, it states that if some x in A is given by equation (2.17), then the series

$$d_1|x_1|^2+\cdots+d_n|x_n|^2+\cdots \tag{2.18}$$

converges.

These results are quite similar to the standard ones for semisimple, commutative H^*-algebras. The H^*-algebras are distinguished by the fact that the two sequences $\{d_i\}$ and $\{D_i\}$ coincide, due to the orthogonality conditions on the minimal idempotents. We are now able to define a continuous homomorphism of an H^*-algebra H_1 into a dense subset of A and a continuous homomorphism of A into an H^*-algebra H_2.

Let H_1 denote an H^*-algebra with a sequence of minimal selfadjoint idempotents $\{s_i\}$ satisfying the conditions that

$$\|s_i\|^2 = D_i.$$

If

$$x = x_1s_1+\cdots+x_ns_n+\cdots \tag{2.19}$$

denotes any element of H_1, then

$$\|x\|^2 = |x_1|^2 D_1+\cdots+|x_n|^2 D_n+\cdots.$$

The multiplication in H_1 satisfies a condition of the form

$$\|xy\| \leqq K'\|x\| \|y\| \tag{2.20}$$

for every $x, y \in H_1$. The mapping T which makes the element x of equation (2.19) correspond to

$$T(x) = x_1e_1+\cdots+x_ne_n+\cdots$$

is a continuous homomorphism of H_1 into A. Since the Im T includes all elements of the form, $x_1e_1+\cdots+x_ne_n$, the Im T is dense in A. The construction of H_2 should be clear.

An equation such as equation (2.20) implies that $1/K' \leqq \|e\|$ for an idempotent e of an algebra A. However, there is no result in the theory of H^*-algebras bounding the norms of the minimal selfadjoint idempotents of the algebra. Nevertheless, an H^*-algebra is somewhat pathological when there is no uniform bound for these. Call a sequence $\{x_n\}$ *admissible* with respect to the basis $\{v_i\}$ if $x_1v_1+\cdots+x_nv_n+\cdots$ converges. For H^*-algebras with bases of uniformly bounded minimal idempotents $\{v_i\}$, the sequence $\{x_n\}$ and $\{x_{p(n)}\}$ are either both admissible or both inadmissible, for any permutation p of the integers. An algebra A with a basis of minimal idempotents $\{e_i\}$ is *homogeneous* if it enjoys this property. I. M. Gel'fand [4] has proved that a homogeneous basis is a Fischer-Riesz basis. Thus we have established most of the following theorem. The remainder of the proof is given in §4.

THEOREM 2.9. *A well-separated, regular algebra A in which every proper closed* *ideal is contained in a maximal modular ideal is an isomorphic image of a homogeneous H^*-algebra if and only if A is homogeneous.*

We close this section with a characterization of regular algebras in terms of their structure space. We recall that for a commutative Banach algebra the structure space is the set of all maximal modular ideals M. Recall that the *hull* $h(I)$ of an ideal I is the set of all maximal modular ideals containing I. The *kernel* $k(S)$ of a set S of maximal modular ideals is their intersection, a closed ideal. The set M of maximal modular ideals is topologized by defining the closure $\bar{S}$ of a set S of M to be the hull of the kernel of S.

THEOREM 2.10. *An algebra A is regular if and only if the structure space M of A is discrete in the hull-kernel topology.*

Proof. Let M denote the structure space of a regular algebra A. Denote by S_i the subset of M consisting of all maximal regular ideals other than R_i. Let J_i be $k(S_i)$, the kernel of S_i. Since (e_i, g_j) is zero for j different from i, it follows that J_i contains the minimal ideal Ae_i. Thus

$$J_i = Ae_i + J_i \cap R_i$$

where the sum is direct. However, if x is an element of $J_i \cap R_i$, then x belongs to the radical and must be zero. Thus J_i coincides with the minimal ideal Ae_i. On the other hand, since (e_i, g_j) is zero only for j not equal to i, it follows that $h(J_i)$ must be S_i.

Consequently,

$$S_i = h(J_i) = h(k(S_i)),$$

and S_i is closed in the hull-kernel topology. Therefore R_i, the complement of S_i, must be open. Since R_i is maximal, it follows that

$$R_i = h(R_i) = h(k(R_i))$$

so that R_i is also closed. The structure space M is discrete.

Suppose, on the other hand, that the structure space of a semisimple, commutative H-algebra A is discrete. Using the notation S_i as above, we see that S_i is closed so that

$$S_i = h(k(S_i)).$$

Consequently, there exists a nonzero vector x_i in $k(S_i)$ which is not in R_i. It follows that (x_i, g_i) is not zero. For each i, define e_i to be $x_i/(x_i, g_i)$. Then

$$(2.21)\qquad \begin{aligned}(e_i, g_j) &= 1, \qquad i = j,\\ &= 0, \qquad i \neq j.\end{aligned}$$

Thus the sequences $\{e_i\}$, $\{g_i\}$, $(i=1, 2, \ldots)$, form a biorthogonal system. We note that $(e_i^2 - e_i, g_j)$ is zero for every j which implies that

$$e_i^2 - e = 0, \qquad \text{or} \qquad e_i^2 = e_i.$$

Consequently, the sequence $\{e_i\}$ consists entirely of idempotents. Furthermore, note that for every x

$$(e_i x, g_j) = (e_i, g_j)(x, g_j).$$

This equation implies that $e_i x$ is an element of R_j whenever i differs from j. To the contrary, $e_i x$ belongs to R_i only when it is the zero vector. In addition, for any x in H,

$$(x - e_i x, g_i) = 0.$$

Hence A is the direct sum $R_i + Ae_i$, i.e., every maximal modular ideal R_i is complemented and A is regular.

3. **Adjoint algebras.** Let H be a Hilbert space on which there exist two binary operations f and f^*. The *standard product* of $x, y \in H$ is the image $f(x, y)$ and is denoted by xy. The *adjoint product* is the image $f^*(x, y)$ denoted by $x \cdot y$ to distinguish it from the standard product. Assume further that constants K and K^* exist such that

$$\|xy\| \leqq K\|x\|\,\|y\|, \qquad \|x\cdot y\| \leqq K^*\|x\|\,\|y\|. \tag{3.01}$$

In addition, there exists a bijection π on H with $\pi(x)$ denoted by x^* such that $\|x^*\| \leqq b\|x\|$, $\|x\| \leqq b^*\|x^*\|$ for all x. The set H is said to form an *adjoint algebra A* if it is a Hilbert algebra under each of the binary operations and the following equation is satisfied for all $x, y, z \in H$

$$(xy, z) = (y, x^* \cdot z). \tag{3.02}$$

We limit our considerations to the case where H is a semisimple, commutative H-algebra under each of the binary operations. We do not repeat these assumptions in the following paragraphs. We use the adjectives, *standard* and *adjoint*, in the expected way. A *standard ideal I* of A is a vector subspace which is closed under standard multiplication by any element in A. An *adjoint maximal modular ideal* is a maximal modular ideal under the adjoint product. Given that each of the products is semisimple, the following rules are valid

$$\begin{aligned} (\varepsilon_1 x + \varepsilon_2 y)^* &= \bar{\varepsilon}_1 x^* + \bar{\varepsilon}_2 y^*, \\ (xy)^* &= y^* \cdot x^* = x^* \cdot y^* \end{aligned} \tag{3.03}$$

where x, y belong to H; ε_1, ε_2 are complex numbers.

The following lemma is clear.

LEMMA 3.1. *The orthogonal complement of a standard (adjoint) ideal I of the adjoint algebra A is an adjoint (standard) ideal J of A.*

LEMMA 3.2. *An adjoint algebra A is regular under each of its two products.*

Proof. Let R be a standard maximal modular ideal of A. Then A is the orthogonal direct sum $R + J$ where J is a one-dimensional adjoint ideal of A. It follows that J is minimal and contains an idempotent generator e^* such that J and $A\cdot e^*$ coincide. Let e denote $\pi^{-1}(e^*)$ so that e is a standard idempotent by equation (3.03). For $y\in R$, $x\in A$,

$$(ye, x) = (y, x\cdot e^*) = 0$$

so that ye is zero for every element $y\in R$. On the other hand, if ye is zero then $(y, x\cdot e^*)=(ye, x)=0$ for every x which implies that $y\in R$, i.e., R is the annihilator of e. Finally, $x=(x-xe)+xe$ for every x in A, that is

$$A = R+Ae \tag{3.04}$$

is a direct sum decomposition of A. Consequently, the standard maximal modular ideal R is complemented and A is regular with respect to the standard product. A similar argument is valid for the adjoint product.

Let $\{e_i\}$ $(i=1, 2, \ldots)$ denote the complete set of standard minimal idempotents of the adjoint algebra A. Then the set $\{e_i^*\}$ $(i=1, 2, \ldots)$ where $e_i^*=\pi(e_i)$, is a complete set of minimal adjoint idempotents of A. Let $\{g_i\}$, $(i=1, 2, \ldots)$, be the set of associated functionals for the set $\{e_i^*\}$. Note that

$$\delta_j^i = (e_i^*, g_i) = (e_i^*, e_i g_i) = (g_i^*, e_i)$$

which implies that the set $\{g_i^*\}$ where $g_i^*=\pi(g_i)$ is the set of associated functions for the set $\{e_i\}$. In this section, we use the symbols $\{e_i\}$, $\{g_i^*\}$ and $\{e_i^*\}$, $\{g_i\}$, $(i=1, 2, \ldots)$, to denote the biorthogonal sequences of minimal idempotents and associated linear functions in the cases of the standard and adjoint operations, respectively, of the adjoint algebra A.

LEMMA 3.3. *The standard (adjoint) socle is dense in the adjoint algebra A.*

Proof. The closed linear hull $[e_i]$, $(e_i\in A)$, is a standard ideal I of A. If the standard socle is not dense, then the orthogonal complement J of I is a proper adjoint ideal of A and

$$A = I+J. \tag{3.05}$$

Let $y\in A$, $x\in J$, and e_i^* be any minimal adjoint idempotent. Then

$$(y, e_i^*\cdot x) = (e_i y, x) = 0 \tag{3.06}$$

so that $e_i^*\cdot x$ is zero for every adjoint minimal idempotent which implies that x is in the adjoint radical. Thus J is a nonzero ideal contained in the adjoint radical, a contradiction. We do not argue the adjoint case since it is essentially the same.

LEMMA 3.4. *Let A be an adjoint algebra such that π is an involution. Then*

$$(x, y) = (y^*, x^*) \tag{3.07}$$

for all $x, y\in A$.

Proof. First, consider the case where one of the elements is a minimal idempotent e_i. Then for any $x \in A$

$$\begin{aligned}(e_i, x) &= (e_i^2, x) = (e_i, e_i^* \cdot x) \\ &= (e_i x^*, e_i^*) = (x^*, e_i^* \cdot e_i^*) = (x^*, e_i^*).\end{aligned}$$

Now let

$$z_n = z_{n1}e_1 + \cdots + z_{nn}e_n$$

and y be any element of A. Then

$$\begin{aligned}(z_n, y) &= (z_{n1}e_1 + \cdots + z_{nn}e_n, y) \\ &= z_{n1}(e_1, y) + \cdots + z_{nn}(e_n, y) \\ &= z_{n1}(y^*, e_1^*) + \cdots + z_{nn}(y^*, e_n^*) \\ &= (y^*, \bar{z}_{n1}e_1^* + \cdots + \bar{z}_{nn}e_n^*) \\ &= (y^*, z_n^*).\end{aligned}$$

We see, in particular, for elements z_n, that

$$\|z_n\|^2 = (z_n, z_n) = (z_n^*, z_n^*) = \|z_n^*\|^2. \tag{3.08}$$

Since the socle is dense, the stated result follows via continuity of the mapping π and of the inner product.

LEMMA 3.5. *Let e_i be a minimal idempotent of the adjoint algebra A. Then*

$$e_i e_i^* = \varepsilon_i e_i \quad \text{and} \quad e_i = \pi_i g_i$$

where ε_i and π_i are complex numbers different from zero.

Proof. First, observe that

$$(e_i, e_j^*) = (e_i e_i, e_j^*) = (e_i, e_i^* \cdot e_j^*)$$

which is zero whenever i differs from j. However, if (e_i, e_i^*) is also zero, then e_i itself must be zero, an impossibility. It follows that e_i must be a nonzero multiple of g_i, that is

$$e_i = \pi_i g_i. \tag{3.09}$$

Furthermore, if $e_i e_i^*$ is zero, then

$$0 = (e_i e_i^*, e_i^*) = (e_i^*, e_i^* \cdot e_i^*) = (e_i^*, e_i^*)$$

so that e_i^* is zero, contradicting its definition as an adjoint minimal idempotent. Thus

$$e_i e_i^* = \varepsilon_i e_i \tag{3.10}$$

where ε_i is not zero.

We wish to observe that in the case of an H^*-algebra, the minimal idempotent e_i is selfadjoint and equation (3.10) holds with ε_i equal to one. An adjoint algebra A is called *normal* if each ε_i satisfying equation (3.10) is positive.

THEOREM 3.6. *Let A be a normal adjoint algebra in which the minimal idempotents are bounded in norm by B. Then A has a basis* $\{e_i\}$ $(i=1, 2, \ldots)$ *of minimal idempotents which is a Fischer-Riesz system.*

Proof. It follows from equations (3.09) and (3.10) that

$$\begin{aligned} \varepsilon_i &= (\varepsilon_i e_i, g_i^*) = (e_i e_i^*, g_i^*) \\ &= (e_i^*, g_i^*) = \bar{\pi}_i(g_i^*, g_i^*) \end{aligned}$$

which proves that π_i is real and positive.

The following inequalities are valid in A:

$$\begin{aligned} 1/B &\leqq \|g_i\| \leqq K, \\ 1/B &\leqq \|g_i^*\| \leqq K^*, \\ 1/K &\leqq \|e_i\| \leqq B, \\ 1/K^* &\leqq \|e_i^*\| \leqq B, \\ 1/K^2 &\leqq \pi_i \leqq B^2. \end{aligned} \tag{3.11}$$

Now let x be any element in A and denote (x, g_i^*) by x_i. Since the standard socle is dense, there exist sequences

$$\begin{aligned} z_n &= x_{n1}e_1 + \cdots + x_{nn}e_n, \\ z_n^* &= \bar{x}_{n1}e_1^* + \cdots + \bar{x}_{nn}e_n^* \\ &= \pi_1\bar{x}_{n1}g_1^* + \cdots + \pi_n\bar{x}_{nn}g_n^*, \end{aligned} \tag{3.12}$$

such that x is the limit of z_n. It follows from the continuity of the norm and inner product that

$$\lim \|z_n\| = \|x\|, \quad \text{and} \quad \lim x_{nj} = x_j. \tag{3.13}$$

From relations (3.11) and (3.12), we see that

$$\begin{aligned} (1/K)^2(|x_{n1}|^2 + \cdots + |x_{nn}|^2) &\leqq \pi_1|x_{n1}|^2 + \cdots + \pi_n|x_{nn}|^2 \\ &= (z_n, z_n^*) \leqq \|z_n\| \, \|z_n^*\| \leqq b\|z_n\|^2. \end{aligned}$$

Given $\varepsilon > 0$, there exists an integer N such that

$$\|z_n\|^2 \leqq (\|x\|^2 + \varepsilon)^2$$

whenever $n > N$. Thus, for this case,

$$(1/K)^2(|x_{n1}|^2 + \cdots + |x_{nn}|^2) \leqq b(\|x\| + \varepsilon)^2,$$

which implies that, for $v = K^2 b$,

$$(|x_1|^2 + \cdots + |x_n|^2) \leqq v(\|x\| + \varepsilon)^2$$

for every ε. Consequently, we have

$$\begin{aligned} (|x_1|^2 + \cdots + |x_n|^2) &\leqq v\|x\|^2, \\ (|x_1|^2 + \cdots + |x_n|^2 + \cdots) &\leqq v\|x\|^2. \end{aligned} \tag{3.14}$$

Equation (3.14) implies that the set $\{e_i\}$ of minimal idempotents is a Bessel system. It follows from the results of Bary, that the system $\{g_i^*\}$, biorthogonal to it, is a Hilbert system. Therefore, given any sequence $\{c_i\}$ in l_2, the sequence $\{z_n\}$ given by

$$z_n = c_1 g_1^* + \cdots + c_n g_n^* \tag{3.15}$$

converges to an element x of A. Furthermore, there exists a constant m such that

$$\|x\| \leqq m(|c_1|^2 + \cdots + |c_n|^2 + \cdots).$$

Note that the sequence $\{c_i\}$ belongs to l_2 if and only if the sequence $\{c_i \pi_i\}$ belongs to l_2. Consequently, if $\{c_i\}$ is in l_2, then the sequence $\{z_n\}$ where

$$\begin{aligned} z_n &= c_1 e_1^* + \cdots + c_n e_n^* \\ &= c_1 \pi_1 g_1^* + \cdots + c_n \pi_n g_n^* \end{aligned} \tag{3.16}$$

converges to an element x in A. In addition,

$$\begin{aligned} \|x\| &\leqq m(|\pi_1 c_1|^2 + \cdots + |\pi_n c_n|^2 + \cdots) \\ &\leqq mB^4(|c_1|^2 + \cdots + |c_n|^2 + \cdots) \end{aligned}$$

Thus, we see that the system $\{e_i^*\}$ is a Hilbert system. However, $\{e_i^*\}$ is a Bessel system by the same sort of argument that proved that $\{e_i\}$ is a Bessel system. It follows that $\{e_i^*\}$ and, consequently, $\{e_i\}$ are Fischer-Riesz systems, as was to be shown.

We turn to one further variation in the axioms in which the results follow almost immediately.

Let A be a commutative, semisimple H-algebra with an involution π where the image $\pi(x)$ is denoted by x^*. Suppose that f and f^* are two inner products on A with $f(x, y)$ and $f^*(x, y)$ denoted by (x, y) and $[x, y]$ respectively. In addition, assume there exist positive constants B and B^* such that

$$|(x, y)|^2 \leqq B^2[x, x][y, y], \quad \text{and} \quad |[x, y]|^2 \leqq (B^*)^2(x, x)(y, y). \tag{3.17}$$

The algebra A is called a *dual adjoint* algebra if the involution and the inner products satisfy the relations

$$(xy, z) = [x, zy^*] = [y, x^*z]. \tag{3.18}$$

THEOREM 3.7. *The dual adjoint algebra A has a basis $\{e_i\}$, $(i=1, 2, \ldots)$, of minimal idempotents. Furthermore, the basis $\{e_i\}$ is a Fischer-Riesz system if and only if the minimal idempotents are uniformly bounded.*

Proof. We define a new inner product on A under which it becomes an H^*-algebra. For $x, y \in A$, let

$$\{x, y\} = (x, y) + [x, y]. \tag{3.19}$$

Denote the norm of x in this new inner product by $|||x|||$. It is easy to see that

$$(3.20)\qquad \begin{aligned} |\{x, y\}| &\leqq (1+B^*)\, \|x\|\, \|y\|, \quad \text{and} \\ |(x, y)| &\leqq \|x\|\, \|y\| \leqq |||x|||\, |||y||| \end{aligned}$$

where $\|x\|^2$ denotes (x, x). Furthermore, if $\|xy\| \leqq K\|x\|\,\|y\|$, then $|||xy||| \leqq K'|||x|||\,|||y|||$ where K' denotes $K(1+B^*)^{1/2}$. Finally,

$$\{xy, z\} = \{y, x^*z\} = \{x, zy^*\}$$

so that A is a semisimple, commutative H^*-algebra in this new norm. We denote the "two" normed algebras by $A(\ ,\ )$ and $A\{\ ,\ \}$ respectively.

The fundamental theorem on H^*-algebras asserts that $A\{\ ,\ \}$ has an orthonormal basis $\{e_i\}$, $(i=1, 2, \ldots)$, of minimal selfadjoint idempotents. The identity mapping of $A\{\ ,\ \}$ onto $A(\ ,\ )$ is a continuous automorphism. It follows that the set $\{e_i\}$ is a basis of minimal idempotents in the algebra $A(\ ,\ )$, the original dual adjoint algebra. It is easy to see that $\{e_i\}$ is a Fischer-Riesz basis in each algebra if and only if the minimal idempotents are uniformly bounded.

4. Examples. The following examples were suggested by the results of N. Bary [3] in which the most natural generalization of orthonormal base is developed. Her more important results for biorthogonal systems $\{e_i\}$, $\{g_i\}$, $(i=1, 2, \ldots)$, in the separable Hilbert space H are in terms of three basic concepts. The complete system $\{e_i\}$ is said to be a *Bessel system* if given any $x \in H$, the coefficients $\{(x, g_i)\}$ belong to l_2. The complete system $\{g_i\}$ is said to be a *Hilbert system* if given any sequence $\{c_i\}$ in l_2 there exists a unique $x \in H$ such that $\{c_i\}$ coincides with the sequence $\{(x, e_i)\}$. A complete system $\{e_i\}$ is said to be a *Fischer-Riesz* system if it is both a Bessel and a Hilbert system. The fundamental results of Bary's are: A system $\{e_i\}$ is a Fischer-Riesz system if and only if it is the image of every orthonormal basis $\{u_i\}$ of H under some continuous automorphism, not necessarily isometric, of H. A system $\{e_i\}$ is a Bessel system if and only if every orthonormal basis $\{u_i\}$ is the image of $\{e_i\}$ under some continuous endomorphism of H. A system $\{e_i\}$ is a Hilbert system if and only if it is the image of every unitary basis $\{u_i\}$ under some continuous endomorphism of H.

The results have the following additional interpretations. Let $\{e_i\}$, $\{g_i\}$ form a biorthogonal system. Let x be in H and consider the following two inequalities:

$$(4.01)\qquad m(|(x, g_1)|^2 + \cdots + |(x, g_n)|^2 + \cdots) \leqq \|x\|^2.$$

$$(4.02)\qquad \|x\|^2 \leqq M(|(x, g_1)|^2 + \cdots + |(x, g_n)|^2 + \cdots).$$

A Fischer-Riesz system $\{e_i\}$ is a basis such that constants m and M exist for which both inequalities (4.01) and (4.02) are valid whenever x belongs to H. Although a complete Hilbert system $\{e_i\}$ need not be a basis, there exists a constant M such that inequality (4.02) is valid for every x in H. Furthermore, a complete Bessel system $\{e_i\}$ need not be a basis, but there exists a constant m such that inequality (4.01) is valid for every x in H.

We need the following Lemmas before considering our examples.

Lemma 4.1. *Let the Hilbert space H be the direct sum $I+J$ of the nonorthogonal, closed subspaces I and J. Let P be the projection of H on I along J. Then if $u \in I$, $v \in J$*

$$|(u, v)| \leqq k\|u\| \, \|v\|, \quad \text{and} \quad 0 < k < 1. \tag{4.03}$$

Proof. Let u and v be elements of I and J, respectively, each being of unit norm. Since u and v are linearly independent, $|(u, v)| < \|u\| \, \|v\| = 1$.

If r is the vector $[u-(u, v)v]/[1-|(u, v)|^2]$, then it is easy to see that

$$\|P\|^2 \geqq \|r\|^2 = 1/[1-|(u, v)|^2] \geqq 1.$$

From which it follows that

$$|(u, v)|^2 \leqq \{\|P\|^2-1\}/\|P\|^2 < k^2 < 1.$$

For general u and v, we have the desired result:

$$|(u, v)| \leqq k\|u\| \, \|v\|.$$

Whenever T is a continuous automorphism of a Hilbert space H onto itself, T has a continuous inverse T^{-1} such that $1 \leqq \|T\| \, \|T^{-1}\|$.

Lemma 4.2. *Let T be a continuous automorphism of a Hilbert space H onto itself. Then, if $\|T\| \, \|T^{-1}\|$ is one, T is essentially unitary, i.e., there exists a positive number r and a unitary transformation U such that*

$$T = rU.$$

Proof. Suppose there exist vectors x_1, x_2 in H such that

$$\|x_1\| = \|x_2\| = 1, \qquad \|Tx_1\| < \|Tx_2\|.$$

Then

$$1/(\|T^{-1}\|) \leqq \|Tx_1\| < \|Tx_2\| \leqq \|T\|$$

so that

$$1 < \|T\| \, \|T^{-1}\|,$$

a contradiction. Thus

$$\|Tx_1\| = \|Tx_2\| = r^2 > 0$$

whenever x_1 and x_2 are of norm one. It follows that

$$((T/r)x, (T/r)x) = (x, x)$$

for every x, i.e., (T/r) is an isometry and, consequently, must be unitary.

Let $\{e_i\}$, $\{g_i\}$ be a biorthogonal system where both $\{e_i\}$ and $\{g_i\}$ are Fischer-Riesz systems. Let $\{u_i\}$ be an orthonormal basis of H and T be the continuous automorphism of H such that

$$e_i = Tu_i, \quad \text{and} \quad 1/(\|T^{-1}\|) \leqq \|e_i\| \leqq \|T\|. \tag{4.04}$$

The continuous automorphism $T' = (T^*)^{-1}$ has the property that

$$g_i = T'u_i, \quad \text{and} \quad 1/\|T^*\| \leqq \|g_i\| \leqq \|T'\|. \tag{4.05}$$

REMARK 4.3. Now let $R \cup S$ be any partition of the integers Z into disjoint subsets and

$$I' = [u_i], \quad (i \in R); \qquad J' = [u_i], \quad (i \in S);$$
$$I = [e_i], \quad (i \in R); \qquad J = [e_i], \quad (i \in S).$$

Then H is the orthogonal direct sum $I' + J'$ of the subspaces I' and J'. It follows from the fact that T is a continuous automorphism that H is also the direct sum $I + J$.

LEMMA 4.4. *The subspaces I and J are well separated, i.e., there exists a constant k with $0 < k < 1$ such that if $u \in I$, $v \in J$ then $|(u, v)| \leqq k\|u\| \, \|v\|$ for any choice of the partitioning sets R and S.*

Proof. Let P and P' denote the orthogonal projections on the subspaces I' and J' respectively. The projections on I and J are given by

$$E = T(P)T^{-1} \quad \text{and} \quad E' = T(P')T^{-1}$$

from which it follows that each of them is bounded by the constant $\|T\| \, \|T^{-1}\|$. It follows from Lemma 4.1 that if k is any constant such that

$$((\|T\| \, \|T^{-1}\|)^2 - 1)/(\|T\| \, \|T^{-1}\|)^2 < k^2 < 1$$

then $u \in I$, $v \in J$ implies $|(u, v)| \leqq k\|u\| \, \|v\|$.

EXAMPLE 4.5. Let $\{e_i\}$ be a Fischer-Riesz basis for the separable Hilbert space H. If

$$x = x_1 e_1 + \cdots + x_n e_n + \cdots, \qquad y = y_1 e_1 + \cdots + y_n e_n + \cdots$$

are any two elements in H, let the product xy be defined by

$$xy = x_1 y_1 e_1 + \cdots + x_n y_n e_n + \cdots. \tag{4.06}$$

Then it follows from inequalities (4.01) and (4.02) that

$$\begin{aligned} \|xy\| &\leqq M(|x_1 y_1|^2 + \cdots + |x_n y_n|^2 + \cdots) \\ &\leqq M(|x_1|^2 + \cdots + |x_n|^2 + \cdots)\,(|y_1|^2 + \cdots + |y_n|^2 + \cdots) \\ &= (M/m^2)[m(|x_1|^2 + \cdots + |x_n|^2 + \cdots)]\,[m(|x_1|^2 + \cdots + |x_n|^2 + \cdots)] \\ &\leqq (M/m^2)\,\|x\|^2\,\|y\|^2. \end{aligned}$$

Consequently, we see that H becomes a Hilbert algebra A with this definition of product. Denote by R_j the set of all linear combinations

$$x = x_1 e_1 + \cdots + x_n e_n + \cdots$$

where the coefficient x_j is zero. Clearly, R_j is a maximal modular ideal. The algebra A is semisimple since the intersection of all the maximal modular ideals in the set $\{R_i\}$, $(i = 1, 2, \ldots)$, is the zero ideal. The basis $\{e_i\}$ is clearly a maximal set of minimal idempotents. Suppose that R is any maximal modular ideal of A. Since R is closed, there exists an e_j which is not contained in R. From this, it follows that R must be contained in R_j and, consequently, must coincide with R_j. The minimal ideal Ae_j is complementary to R. Thus A is a regular algebra. The sequence $\{g_i\}$

biorthogonal to $\{e_i\}$ is the set of multiplicative elements associated with the set $\{R_i\}$ of maximal modular ideals. Given that A is the direct sum $I+J$ of two proper closed ideals, then I coincides with $[e_i]$, $(e_i \in I)$, and J coincides with $[e_i]$, $(e_i \in J)$. It follows from Lemma 4.4 that A is well separated. Inequalities (4.01) and (4.02) imply that A is homogeneous. Thus we see that A is an example of a homogeneous, semisimple, commutative well-separated regular algebra in which every proper closed ideal is contained in a maximal regular ideal.

Furthermore, the mapping T^{-1} of equation (4.04) is an isomorphism of A onto an H^*-algebra H generated by taking the orthonormal basis $\{u_i\}$ as the selfadjoint minimal idempotents of H. Consequently, every H-algebra A with a set $\{e_i\}$ of minimal idempotents which form a Fischer-Riesz basis for the space A is isomorphic to an H^*-algebra. This result shows in particular that a homogeneous, well-separated, regular algebra A in which every proper closed ideal is contained in a maximal regular ideal is isomorphic to an H^*-algebra. When the isomorphism is also isometric, it is easy to see as well that the original algebra is actually an H^*-algebra and the isomorphism is a *-isomorphism.

We note that the definition of the multiplication can be framed somewhat differently. If $x, y \in A$, then the product xy can be defined to be that unique element z in A such that

$$(z, g_i) = (x, g_i)(y, g_i). \tag{4.07}$$

A second product $x \cdot y$ can be defined to be that unique element z' in A such that

$$(z', e_i) = (x, e_i)(y, e_i). \tag{4.08}$$

This last definition is equivalent to defining $x \cdot y$ for

$$x = x_1 g_1 + \cdots + x_n g_n + \cdots,$$
$$y = y_1 g_1 + \cdots + y_n g_n + \cdots$$

to be given by

$$x \cdot y = x_1 y_1 g_1 + \cdots + x_n y_n g_n + \cdots.$$

A sequence $\{x_n\}$ is admissible for the Fischer-Riesz basis $\{e_i\}$ if and only if it is admissible for the system $\{g_i\}$. Consequently, the mapping π defined by

$$\pi(x_1 e_1 + \cdots + x_n e_n + \cdots) = \bar{x}_1 g_1 + \cdots + \bar{x}_n g_n + \cdots \tag{4.09}$$

is a bijection on A. Note that π is usually not an involution; however, if we denote $\pi(x)$ by x^*, then

$$(xy, z) = (x, z \cdot y^*) = (y, x^* \cdot z) \tag{4.10}$$

so that A is an example of an adjoint algebra.

We briefly consider an algebra A which is a continuous automorphic image of an H^*-algebra H whose minimal idempotents $\{s_i\}$ are uniformly bounded by B. If T is the given automorphism, then T maps the set of all minimal selfadjoint idempotents $\{s_i\}$ of H onto the set of all minimal idempotents $\{e_i\}$ of A. It is easy to see that $\{e_i\}$ is a Fischer-Riesz system so that all of the results on algebras with a Fischer-Riesz basis of minimal idempotents apply to A. It should be clear now that if A is an algebra with a Fischer-Riesz basis of minimal idempotents, then A is a continuous isomorphic image of any H^*-algebra H whose minimal idempotents are uniformly bounded. This statement finishes the proof of Theorem 2.9 and completes our discussion of this particular kind of algebra.

We end with a counterexample:

EXAMPLE 4.6. Let A be an H^*-algebra whose maximal set of selfadjoint minimal idempotents $\{s_i\}$ has the property that $\|s_k\|$ is equal to k. It is easy to see that A is a well-separated, regular algebra in which every proper closed ideal is contained in a maximal regular ideal. Nevertheless, it is clear that there exists no continuous automorphism carrying a unitary basis $\{u_i\}$ onto the set of minimal idempotents $\{s_i\}$. Consequently, the set $\{s_i\}$ is not a Fischer-Riesz basis and A is not homogeneous.

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